JINR E2-95-250

On the structure of the $N = 4$ Supersymmetric Quantum Mechanics in $D = 2$ and $D = 3$

V. Berezovoj*

Institute of Physics and Technology
310108, Kharkov, Ukraine
and

A. Pashnev†

JINR–Laboratory of Theoretical Physics
Dubna, Head Post Office, P.O.Box 79, 101 000 Moscow, Russia

Abstract

The superfield formulation of two - dimensional $N = 4$ Extended Supersymmetric Quantum Mechanics (SQM) is described. It is shown that corresponding classical Lagrangian describes the motion in the conformally flat metric with additional potential term. The Bose and Fermi sectors of two- and three-dimensional $N = 4$ SQM are analyzed. The structure of the quantum Hamiltonians is such, that the usual Shrödinger equation in the flat space arises after some unitary transformation, demonstrating the effect of transmutation of the coupling constant and the energy of the initial model in some special cases.

Submitted to "Classical and Quantum Gravity"

*BITNET: KFTI%KFTI.KHARKOV.UA@RELAY.USSR.EU.NET
†BITNET: PASHNEV@THSUN1.JINR.DUBNA.SU

1 Introduction

The simplest way to construct the classical Lagrangian of the SQM in D dimensions is to consider superfields $\Phi^i(\tau, \eta^a),\ i = 1, 2, ...D$ in the superspace $\tau, \eta^a\ a = 1, 2, ...N$ with one bosonic and N grassmann coordinates. The first component of the superfields are the usual bosonic coordinates X^i , the next ones ψ^{ia} are grassmann coordinates. So, the classical Lagrangian of the Supersymmetric Quantum Mechanics (SQM) describes the evolution of bosonic and additional grassmann degrees of freedom, which after quantization become generators of the Clifford algebra. This fact naturally leads to the matrix realization of the Hamiltonian and Supercharges of SQM [1].

The dimensionality of such realization depends on the number of grassmann variables and in the case of scalar superfields Φ^i it is $2^{[\frac{DN}{2}]}$. So, it rapidly growths for extended supersymmetry and, for example, takes the value $d = 64$ for $D = 3,\ N = 4$ case. The way out of this difficulty is to use more complicated representations of extended supersymmetry [2] - [6]. The main idea The simplest of them is the chiral superfield, which contains one complex bosonic and $\frac{N}{2}$ complex grassmann fields. The Lagrangian for such superfield naturally describes the two - dimensional SQM. The main idea of the reduction of the number of grassmann degrees of freedom for other values of D is to use superfields Φ^i which transform nontrivially under the isomorphism algebra of the extended supersymmetry algebra

$$\{Q^a, Q^b\} = i\delta^{ab}\frac{\partial}{\partial\tau}. \tag{1.1}$$

If $N = 4$ the isomorphism algebra is $SO(4) = SO(3) \times SO(3)$ and index i can describe three-dimensional vector representation of one of the $SO(3)$ algebras which plays simultaneously the role of space rotations algebra. The dimensionality of the matrix representation is $d = 4$. The Hamiltonian and Supercharges for such description of three-dimensional $N = 4$ SQM were obtained in [6]. In the present paper we describe the two-dimensional $N = 4$ SQM with the help of the chiral superfield. Along with two dynamic bosonic coordinates such superfield contains four dynamic grassmann coordinates and dimensionality of the matrix representation for Hamiltonian and Supercharges is four again.

In the third and forth sections we analyze Bose and Fermi sectors of the two-dimensional and three-dimensional SQM in such minimal superfield approach. Despite of the complicated structure of the gravitation-like interaction with additional nontrivial potential energy, the resulting stationary equations after some unitary transformation become the usual stationary Shrödinger equations, demonstrating the coupling constant - energy transmutation in some cases.

2 Two - dimensional SQM

In this section we construct the two - dimensional $N = 4$ SQM in the frames of the superfield approach. The $N = 4$ supersymmetry algebra can be written in terms of two complex supercharges $Q_a,\ a = 1, 2$ and hamiltonian H

$$\{Q_a, \overline{Q}^b\} = \delta^b_a H,\ [H, Q_a] = [H, \overline{Q}^a] = 0,\ \overline{Q}^a = Q^\star_a. \tag{2.1}$$

Its automorphism group is $SO(4) = SU(2) \times SU(2)$ and Q_a transforms as a spinor of one of the $SU(2)$ groups.

The chiral superfield [1]

$$\Phi(\tau, \theta, \overline{\theta}) = Z(\tau) + \theta^a \chi_a(\tau) + \frac{i}{2}\overline{\theta}\theta \dot{Z}(\tau) + \theta\theta F(\tau) - \frac{i}{4}\theta\theta\overline{\theta}_a \dot{\chi}^a - \frac{1}{16}\overline{\theta\theta}\theta\theta \ddot{Z}(\tau) \tag{2.2}$$

in the superspace with one bosonic coordinate τ and two complex fermionic coordinates θ^a behaves as a scalar under the supersymmetry transformations

$$\delta\theta^a = \epsilon^a, \ \delta\overline{\theta}_a = \overline{\epsilon}_a, \ \delta\tau = \frac{i}{2}(\epsilon^a \overline{\theta}_a + \overline{\epsilon}_a \theta^a) \tag{2.3}$$

and satisfies the chirality conditions $\overline{D}^a \Phi = 0$, where

$$\overline{D}^a = \frac{\partial}{\partial \overline{\theta}_a} - \frac{i}{2}\theta^a \frac{\partial}{\partial \tau} \tag{2.4}$$

are the supersymmetric covariant derivatives.

The most general action for the superfield Φ

$$S = \frac{1}{2}\int d\tau d^2\theta d^2\overline{\theta} V(\Phi, \overline{\Phi}) + \int d\tau d^2\theta R(\Phi) + \int d\tau d^2\overline{\theta}\overline{R}(\overline{\Phi}). \tag{2.5}$$

contains one real function $V(\Phi, \overline{\Phi})$ and one chiral function $R(\Phi)$. After the integration over θ and $\overline{\theta}$ we find the component Lagrangian, in which fields F and $\overline{F}$ are auxiliary and they can be dropped with the help of their equations of motion.

Finally, the component Lagrangian takes form:

$$\begin{aligned} L \ &= \ \frac{1}{2}W\dot{z}\dot{\overline{z}} + i(\psi^a \dot{\overline{\psi}}_a) - \frac{i}{W}(\dot{z}\frac{\partial W}{\partial z} - \dot{\overline{z}}\frac{\partial W}{\partial \overline{z}})\psi\overline{\psi} - \frac{\partial}{\partial z}\left(\frac{U}{W}\right)\psi\psi - \frac{\partial}{\partial \overline{z}}\left(\frac{\overline{U}}{W}\right)\overline{\psi\psi} + \\ &+ \frac{2}{W^2}(\frac{\partial^2 W}{\partial z \partial \overline{z}} - \frac{1}{W}\frac{\partial W}{\partial z}\frac{\partial W}{\partial \overline{z}})\psi\psi\overline{\psi\psi} - \frac{U\overline{U}}{2W}, \end{aligned} \tag{2.6}$$

where we have introduced

$$W(z, \overline{z}) = \frac{\partial^2 V(z, \overline{z})}{\partial z \partial \overline{z}}, \ \ U(z) = \frac{\partial R(z)}{\partial z}, \ \ \psi = \frac{\sqrt{W}}{2}\chi. \tag{2.7}$$

The quantization procedure takes into account the dependence of the Lagrangian on the conformally flat metric $g_{ik}\dot{z}^i\dot{z}^k = W(z, \overline{z})\dot{z}\dot{\overline{z}}$ [6] and leads to the following quantum Hamiltonian and Supercharges:

$$\begin{aligned} \hat{H} \ &= \ \frac{2}{W}\mathcal{P}\overline{\mathcal{P}} + \frac{2i}{W^2}(\frac{\partial W}{\partial z}\overline{\mathcal{P}} - \frac{\partial W}{\partial \overline{z}}\mathcal{P})(\frac{1}{2} - \overline{\psi}\psi) + \\ & \quad \frac{4}{W^2}(\frac{3}{2W}\frac{\partial W}{\partial z}\frac{\partial W}{\partial \overline{z}} - \frac{\partial^2 W}{\partial z \partial \overline{z}})(\frac{1}{2} - \overline{\psi}\psi)^2 + \frac{\partial}{\partial z}\left(\frac{U}{W}\right)\psi\psi + \frac{\partial}{\partial \overline{z}}\left(\frac{\overline{U}}{W}\right)\overline{\psi\psi} + \frac{U\overline{U}}{2W}, \end{aligned} \tag{2.8}$$

$$\hat{Q}^a \ = \ (\pi\psi^a + i\frac{\overline{U}}{2}\overline{\psi}^a)\frac{2}{\sqrt{W}}, \tag{2.9}$$

$$\hat{\overline{Q}}_a \ = \ \frac{2}{\sqrt{W}}(\overline{\psi}_a \overline{\pi} + i\frac{U}{2}\psi_a), \tag{2.10}$$

[1] Our conventions for spinors are as follows: $\overline{\theta}_a \equiv (\theta^a)^\star, \theta_a \equiv \theta^b \varepsilon_{ba}, \theta^a = \varepsilon^{ab}\theta_b, \overline{\theta}^a \equiv \varepsilon^{ab}\overline{\theta}_b, \overline{\theta}_a = \overline{\theta}^b \varepsilon_{ba}, \overline{\theta}^a = -(\theta_a)^\star, (\theta\theta) \equiv \theta^a\theta_a = -2\theta^1\theta^2, (\overline{\theta\theta}) \equiv \overline{\theta}_a\overline{\theta}^a = (\theta\theta)^\star$.

where

$$\pi=\mathcal{P}+\frac{i}{W}\frac{\partial W}{\partial z}(\psi\overline{\psi}-\frac{3}{2}),\overline{\pi}=\overline{\mathcal{P}}-\frac{i}{W}\frac{\partial W}{\partial \overline{z}}(\psi\overline{\psi}-\frac{1}{2}), \tag{2.11}$$

and $\mathcal{P}=-i\frac{\partial}{\partial z}$, $\overline{\mathcal{P}}=-i\frac{\partial}{\partial \overline{z}}$. Grassmann variables $\overline{\psi}_b$ and ψ^a satisfy the following commutation relations

$$\{\psi^a,\overline{\psi}_b\}=\frac{1}{2}\delta^a_b,\ a,b=1,2\ . \tag{2.12}$$

and can be considered as a creation and annihilation operators. The general quantum state can be written as the vector in the corresponding Fock space

$$|\Phi(z,\overline{z})\rangle=\phi^1(z,\overline{z})|0\rangle+\chi^a(z,\overline{z})\overline{\psi}_a|0\rangle+\phi^2(z,\overline{z})\overline{\psi}^a\overline{\psi}_a|0\rangle, \tag{2.13}$$

where $|0\rangle$ is the vacuum of the Fock space: $\psi^a|0\rangle=0$. The Hamiltonian $\hat{H}$ is hermitian and hermiticity properties of supercharges $\overline{\hat{Q}^a}=\hat{Q}^+_a$ are fulfilled with respect to the scalar product

$$(\Phi_1,\Phi_2)=\int d^2zW(z,\overline{z})\langle\Phi_1(z,\overline{z})|\Phi_2(z,\overline{z})\rangle. \tag{2.14}$$

The only case when the scalar product (3.17) coincides with the usual scalar product is $W(z,\overline{z})=const$.

The normalizable solutions of the stationary Schrödinger equation

$$\hat{H}|\Phi\rangle=E|\Phi\rangle \tag{2.15}$$

describe the physical states. In Bose sector this equation is non diagonal and leads to two connected equations for wavefunctions $\phi^A(z,\overline{z})$

$$\{\frac{2}{W}\mathcal{P}\overline{\mathcal{P}}+\frac{i}{W^2}(\frac{\partial W}{\partial z}\overline{\mathcal{P}}-\frac{\partial W}{\partial \overline{z}}\mathcal{P})+\frac{1}{W^2}(\frac{3}{2W}\frac{\partial W}{\partial z}\frac{\partial W}{\partial \overline{z}}-\frac{\partial^2 W}{\partial z\partial\overline{z}})+\frac{U\overline{U}}{2W}\}\phi^1(z,\overline{z})+$$
$$\frac{\partial}{\partial\overline{z}}\left(\frac{\overline{U}}{W}\right)\phi^2(z,\overline{z})=E\phi^1(z,\overline{z}) \tag{2.16}$$
$$\{\frac{2}{W}\mathcal{P}\overline{\mathcal{P}}-\frac{i}{W^2}(\frac{\partial W}{\partial z}\overline{\mathcal{P}}-\frac{\partial W}{\partial \overline{z}}\mathcal{P})+\frac{1}{W^2}(\frac{3}{2W}\frac{\partial W}{\partial z}\frac{\partial W}{\partial \overline{z}}-\frac{\partial^2 W}{\partial z\partial\overline{z}})+\frac{U\overline{U}}{2W}\}\phi^1(z,\overline{z})+$$
$$\frac{\partial}{\partial z}\left(\frac{U}{W}\right)\phi^2(z,\overline{z})=E\phi^2(z,\overline{z}) \tag{2.17}$$

It means that true physical states in the case of nonvanishing chiral terms in the action (2.5) are superpositions of the bosonic states with different fermionic number (0 and 2). After the unitary transformation

$$\hat{H}=W^{-1/2}\hat{\tilde{H}}W^{1/2},\quad \phi^A=W^{-1/2}\tilde{\phi}^A \tag{2.18}$$

the system (2.16)-(2.17) takes more simple form

$$\{2\mathcal{P}\frac{1}{W}\overline{\mathcal{P}}+\frac{U\overline{U}}{2W}\}\tilde{\phi}^1(z,\overline{z})+\frac{\partial}{\partial\overline{z}}\left(\frac{\overline{U}}{W}\right)\tilde{\phi}^2(z,\overline{z})\ =\ E\tilde{\phi}^1(z,\overline{z}) \tag{2.19}$$

$$\{2\overline{\mathcal{P}}\frac{1}{W}\mathcal{P}+\frac{\alpha\overline{\alpha}}{8W}\}\tilde{\phi}^2(z,\overline{z})+\frac{\partial}{\partial z}\left(\frac{U}{W}\right)\tilde{\phi}^1(z,\overline{z})\ =\ E\tilde{\phi}^2(z,\overline{z}). \tag{2.20}$$

There is no need to solve this system. The simplest way to find the solution of this system is to find the solution of the system in the sector with fermionic number 1 and to apply one of the supercharges Q_a to it. As a consequence of the supersymmetry algebra (2.1) it will be the solution in the bosonic sector.

In the Fermi sector the Hamiltonian $\hat{H}$ is diagonal and equations for the spinors $\chi^a(z,\overline{z})$ are very simple:

$$\frac{2}{W}(\mathcal{P}\overline{\mathcal{P}} + \frac{U\overline{U}}{4})\chi^a(z,\overline{z}) = E\chi^a(z,\overline{z}). \tag{2.21}$$

These equations are just the stationary zero energy Shrödinger equations

$$(2\mathcal{P}\overline{\mathcal{P}} + \tilde{U}(z,\overline{z}))\chi^a(z,\overline{z}) = 0 \tag{2.22}$$

with the potential

$$\tilde{U}(z,\overline{z}) = \frac{U(z,\overline{z})\overline{U(z,\overline{z})}}{2} - EW(z,\overline{z}). \tag{2.23}$$

The whole potential $\tilde{U}$ is combined from two functions $U(z)$ and $W(z,\overline{z})$. Some particular choices of this functions are interesting. The simplest of them $W(z,\overline{z}) = 1$ leads to the standard Shrödinger equation with the potential

$$\tilde{U}(z,\overline{z}) = \frac{U(z,\overline{z})\overline{U(z,\overline{z})}}{4} \tag{2.24}$$

and the energy E. The opposite situation takes place when $U(z) = \alpha = const$ and $W(z,\overline{z})$ is arbitrary. The energy E in the corresponding Shrödinger equation

$$(2\mathcal{P}\overline{\mathcal{P}} - EW(z,\overline{z}))\chi^a(z,\overline{z}) = -\frac{1}{2}\alpha\overline{\alpha}\chi^a(z,\overline{z}) \tag{2.25}$$

plays the role of the coupling constant. In turn, the coupling constant α , namely its function $\mathcal{E} = -\frac{1}{2}\alpha\overline{\alpha}$, plays the role of the energy. This situation demonstrates the effect of coupling constant - energy transmutation. As we will see later, only this situation takes place in three dimensions.

In the general situation of arbitrary $U(z)$ and $W(z,\overline{z})$ the energy E plays the role of the coupling constant and the physical spectrum is determined by the existence of the normalizable solutions of the equation (2.23) at the definite values E_n of the parameter E.

3 Hamiltonian and supercharges in $D = 3$

The following expressions for Hamiltonian and supercharges for three - dimensional SQM have been obtained in [6]:

$$\hat{H} = -\frac{1}{2}\frac{1}{W(x)}\partial_i^2 - \frac{1}{4}\frac{\partial_i W(X)}{W^2(x)}\partial_i + \frac{1}{2}\frac{\alpha^2}{W(x)} - \frac{3}{8}\frac{\partial_i^2 W(x)}{W^2(x)} + \frac{15}{32}\frac{(\partial_i W(x))^2}{W^3(x)} +$$

$$+(\overline{\psi}_a\psi^a - \frac{1}{2})^2\left(\frac{\partial_i^2 W(X)}{W^2(x)} - \frac{3}{2}\frac{(\partial_i W(x))^2}{W^3(x)}\right) + \tag{3.1}$$

$$+i\epsilon_{ikl}\psi\sigma_i\overline{\psi}\frac{\partial_k W(x)}{W^2(x)}\partial_l - \alpha\psi\sigma_i\overline{\psi}\frac{\partial_i W(x)}{W^2(x)},$$

$$\hat{Q}_a = \left[(\sigma_i)_a^b\left(p_i - i(\overline{\psi}_c\psi^c - \frac{1}{2})\partial_i \ln W(x)\right) + i\alpha\delta_a^b\right]\frac{\overline{\psi}_b}{\sqrt{W(x)}}, \tag{3.2}$$

$$\hat{\overline{Q^a}} = \frac{\overline{\psi}^b}{\sqrt{W(x)}}\left[(\sigma_i)_b^a\left(p_i + i(\overline{\psi}_c\psi^c - \frac{1}{2})\partial_i \ln W(x)\right) - i\alpha\delta_a^b\right], \tag{3.3}$$

where i, k, l are three-dimensional vector indices and $a, b = 1, 2$ - spinor indices as in the two - dimensional case. Again the operators $\overline{\psi}_b$ and ψ^a are creation and annihilation operators and the general quantum state can be written as the x-depending vector in the corresponding Fock space

$$|\Phi(x)\rangle = \phi^1(x)|0\rangle + \chi^a(x)\overline{\psi}_a|0\rangle + \phi^2(x)\overline{\psi}^a\overline{\psi}_a|0\rangle, \tag{3.4}$$

In 3 dimensions the Hamiltonian $\hat{H}$ and operators

$$p_i = -i\partial_i - \frac{3}{4}i\partial_i W(x) \tag{3.5}$$

are Hermitian and $\hat{\overline{Q^a}} = \hat{Q}_a^+$ with respect to the scalar product

$$(\Phi_1, \Phi_2) = \int d^3x W^{3/2}(x)\langle\Phi_1(x)|\Phi_2(x)\rangle, \tag{3.6}$$

which contains the measure $W^{3/2}$ in contrast to 2 - dimensional case.

All of the operators (2.1)-(2.3) are completely determined in terms of the function $W(x)$ (connected with the superpotential $V(x)$: $W(x) = \partial_i\partial_i V(x)$) and one additional parameter α, which also characterizes the parity violation. The sum of the angular momentum and spin operators

$$\hat{J}_i = \epsilon_{ikl}x_k p_l + \psi^a(\sigma_i)_a^b\overline{\psi}_b \tag{3.7}$$

is conserved operator, describing the total momentum of the system. The eigenvalues of the operator (3.7) are integer for bosonic states with wavefunctions $\phi^A(x),\ A = 1, 2,$ and half-integer for fermionic states with grassmann spinor wavefunctions $\chi^a(x)$.

The normalizable solutions of the stationary Schrödinger equation

$$\hat{H}|\Phi\rangle = E|\Phi\rangle \tag{3.8}$$

describe the physical states. In Bose sector this equation is diagonal and leads to two identical equations

$$\hat{H}_B\phi^A(x) = E\phi^A(x) \tag{3.9}$$

with[2]

$$\hat{H}_B = -\frac{1}{2} g^{-\frac{1}{2}} \hat{\partial}_i g^{ij}(x) g^{\frac{1}{2}} \partial_j + \frac{\alpha^2}{2W} - \frac{1}{16} R, \tag{3.10}$$

where

$$g_{ij}(x) = W(x)\delta_{ij} \tag{3.11}$$

is the metric tensor and

$$R = 2\left(\frac{\partial_i \partial_i W(x)}{W^2(x)} - \frac{3}{4}\frac{(\partial_i W(x))^2}{W^3(x)}\right) \tag{3.12}$$

- the scalar curvature of the corresponding three dimensional space. Note the equality of the coefficient at R to that, calculated in [7]. In the Fermi sector the Hamiltonian $\hat{H}$ is non diagonal and the equation for the spinor $\chi^a(x)$ is as follows:

$$\begin{aligned}\left\{-\frac{1}{2} g^{-\frac{1}{2}} \hat{\partial}_i g^{ij}(x) g^{\frac{1}{2}} \partial_j + \frac{\alpha^2}{2W} - \frac{3}{8}\left(\frac{\partial_i \partial_i W(x)}{W^2(x)} - \frac{5}{4}\frac{(\partial_i W(x))^2}{W^3(x)}\right)\right\} \chi^a(x) - \\ -\frac{1}{2}\left\{ i\epsilon_{ikl} \frac{\partial_k W(x)}{W^2(x)} \partial_l - \alpha \frac{\partial_i W(x)}{W^2(x)}\right\} (\sigma_i)^a_b \chi^b(x) \quad = \quad \chi^a(x).\end{aligned} \tag{3.13}$$

The first of nondiagonal terms in this equation is proportional to the spin-orbit coupling and the second one leads to the parity violation.

Both the bosonic and fermionic equations can be written in more simple form with the help of transformation

$$\hat{H} = W^{-\frac{1}{4}}(x) \hat{\tilde{H}} W^{\frac{1}{4}}(x),\ \phi^A(x) = W^{-\frac{1}{4}}(x)\tilde{\phi}^A(x),\ \chi^a(x) = W^{-\frac{1}{4}}(x)\tilde{\chi}^a(x). \tag{3.14}$$

The transformed equations are:

$$\left\{-\frac{1}{2}\partial_i^2 - EW(x)\right\}\tilde{\phi}^A(x) \quad = \quad -\frac{1}{2}\alpha^2 \tilde{\phi}^A(x), \tag{3.15}$$

$$\begin{aligned}\left\{-\frac{1}{2}\partial_i^2 - EW(x) + \frac{1}{2} W^{\frac{1}{2}} \partial_i^2 W^{-\frac{1}{2}}\right\}\tilde{\chi}^a(x) - \\ -\frac{1}{2}\left\{ i\epsilon_{ikl}\partial_k \ln W(x) \partial_l - \alpha \partial_i \ln W(x)\right\} (\sigma_i)^a_b \tilde{\chi}^b(x) \quad = \quad -\frac{1}{2}\alpha^2 \tilde{\chi}^a(x).\end{aligned} \tag{3.16}$$

The bosonic ones are just the stationary Shrödinger equations with the potential $U(x) = -EW(x)$. The energy E plays the role of the coupling constant and the coupling constant α , namely its function $\mathcal{E} = -\frac{1}{2}\alpha^2$, plays the role of the energy. It means, that the effect of coupling constant - energy metamorphosis takes place in three-dimensional SQM as well.

In spite of the fact that the bosonic Schrödinger equation (3.15) is written in the flat three dimensional space, the scalar product for wavefunctions $\tilde{\phi}^A(x)$ contains the function $W(x)$:

$$(\phi^A, \phi^B) = \delta^{AB} \int d^3x W(x) \phi^{A*}(x) \phi^B(x). \tag{3.17}$$

This relation is the consequence of the relations (3.6)-(3.14).

[2] to avoid misunderstandings we denote the derivative which acts on everything to the right by means of $\hat{\partial}_i$

4 Example

As an illustration we consider in three dimensions the case $W(x) = \frac{\kappa}{r}$, $r = \sqrt{x_i^2}$. The scalar product is

$$(\phi_1, \phi_2) = \int d^3x \frac{\kappa}{r} \phi_1^*(x) \phi_2(x). \tag{4.1}$$

Taking the standard form for the wavefunction

$$\phi(x) = \frac{1}{r} \sum_{l,m} u_{nl}(r) Y_{lm}(\theta, \phi), \tag{4.2}$$

we find the equation for $u_{nl}(r)$:

$$u_{nl}^{''}(r) + \left\{ \frac{2\kappa E}{r} - \frac{l(l+1)}{r^2} - \alpha^2 \right\} u_{nl}(r) = 0. \tag{4.3}$$

with normalization condition

$$(u_{n'l'}, u_{nl}) = \kappa \int \frac{dr}{r} u_{n'l'}^*(r) u_{nl}(r) = \delta_{n'n} \delta_{l'l}. \tag{4.4}$$

The normalizable solutions of the equation (4.3) are

$$u_{nl}(r) = C_{nl}\, r^l\, e^{-\alpha r}\, {}_1F_1(l+1-n, 2l+2; 2\alpha r) \tag{4.5}$$

with constant C_{nl}. The energy spectrum is given by the following relation

$$E_{nl} = \frac{\alpha}{\kappa}(n+l+1) \tag{4.6}$$

The solutions of the corresponding equation in the Fermi-sector for $W(x) = \frac{\kappa}{r}$

$$\begin{aligned} \left\{ -\frac{1}{2}\partial_i^2 - \frac{E\kappa}{r} + \frac{3}{8}\frac{1}{r^2} \right\} \tilde{\chi}^a(x) + & \\ + \frac{1}{2r^2} \left\{ i\epsilon_{ikl} x_k \partial_l - \alpha x_i \right\} (\sigma_i)^a_b \tilde{\chi}^b(x) &= -\frac{1}{2}\alpha^2 \tilde{\chi}^a(x). \end{aligned} \tag{4.7}$$

can be obtained from the solutions of bosonic equation with the help of supersymmetry transformations. The energy spectrum of fermionic equation (4.7) is also given by the formula (4.6).

Acknowledgments. We would like to thank D.V. Volkov and E.A. Ivanov for useful discussions and comments on the subject. This investigation has been supported in part by the Russian Foundation of Fundamental Research, grant 93-02-03821, the International Science Foundation, grants M9T300 and UA6000 and INTAS, grant 93-127

References

[1] E.Witten, Nucl.Phys. **B188** (1981) 513

[2] V.P.Akulov, A.I.Pashnev, Theor.Math.Phys., **65** (1985) 1027

[3] V.P.Berezovoj, A.I.Pashnev, Preprint KFTI 91-20, Kharkov, (1991)

[4] E.A.Ivanov, S.O.Krivonos, A.I.Pashnev, Class.Quantum Grav. **8** (1991) 19

[5] E.A.Ivanov, A.V.Smilga, Phys.Lett. **B257** (1991) 79

[6] V.P.Berezovoj, A.I.Pashnev. Class.Quant.Grav.,**8** (1991) 2141.

[7] T.Christodoulakis, E.Korfiatis. Phys.Lett.,**B256** (1991) 484.